Chapter 4 Appendix 1 **Convolution in general**

Idea of the Convolution Method in general This concerns $f = g * h$ when neither g nor h is $\equiv 1$. For an example we will look at Euler's phi function $\phi = \mu * j$, where $j(n) = n$ for all n. In general, if $f(n) = \sum_{ab=n} g(a)\, h(b)$, then

$$\sum_{n \le x} f(n) = \sum_{ab \le x} g(a)\, h(b) \,. \tag{10}$$

This can be rearranged as both

$$\sum_{a \le x} g(a) \sum_{b \le x/a} f(b) \quad \text{and} \quad \sum_{b \le x} f(b) \sum_{a \le x/b} g(a) \,.$$

Which one you choose depends on the situation. For example, for $\phi = \mu * j$ it would be inappropriate to start as in

$$\sum_{n \le x} \phi(n) = \sum_{b \le x} j(b) \sum_{a \le x/b} \mu(a) \,,$$

since we have no results on the sum of the Möbius function. In fact bounds on $\sum_{a \le x} \mu(a)$ are as difficult to prove as the Prime Number Theorem, and further, the statement

$$\frac{1}{x} \sum_{a \le x} \mu(a) \to 0 \text{ as } x \to \infty$$

is *equivalent* to the Prime Number Theorem, $\psi(x) \sim x$.

Theorem 4.16

$$\sum_{n \le x} \phi(n) = \frac{x^2}{2\zeta(2)} + O\left(x \log x\right) .$$

Solution start from

$$\sum_{n \le x} \phi(n) = \sum_{a \le x} \mu(a) \sum_{b \le x/a} j(b) \,.$$

For this inner sum

$$\sum_{b \le x/a} j(b) = \sum_{b \le \left[\frac{x}{a}\right]} b = \frac{1}{2}\left[\frac{x}{a}\right]\left(\left[\frac{x}{a}\right] + 1\right) ,$$

having simply summed the first $[x/a]$ integers. Continuing

$$\begin{aligned}
\frac{1}{2}\left[\frac{x}{a}\right]\left(\left[\frac{x}{a}\right]+1\right) &= \frac{1}{2}\left(\frac{x}{a}+O(1)\right)\left(\frac{x}{a}+O(1)+1\right) \\
&= \frac{1}{2}\left(\frac{x}{a}+O(1)\right)\left(\frac{x}{a}+O(1)\right) \\
&= \frac{1}{2}\left(\frac{x^2}{a^2}+O\left(\frac{x}{a}\right)+O(1)\right) \\
&= \frac{x^2}{2a^2}+O\left(\frac{x}{a}\right),
\end{aligned}$$

having kept the largest error term. Substituting back in

$$\begin{aligned}
\sum_{n\le x}\phi(n) &= \sum_{a\le x}\mu(a)\left(\frac{x^2}{2a^2}+O\left(\frac{x}{a}\right)\right) \\
&= \frac{x^2}{2}\sum_{a\le x}\frac{\mu(a)}{a^2}+O\left(x\sum_{a\le x}\frac{1}{a}\right) \\
&= \frac{x^2}{2}\left(\sum_{a=1}^{\infty}\frac{\mu(a)}{a^2}-\sum_{a>x}\frac{\mu(a)}{a^2}\right)+O(x\log x) \\
&\qquad \text{since } \textstyle\sum_{a\le x}1/a=O\left(\log x\right) \\
&= \frac{x^2}{2}\left(\frac{1}{\zeta(2)}+O\left(\sum_{a>x}\frac{1}{a^2}\right)\right)+O(x\log x) \\
&= \frac{x^2}{2}\left(\frac{1}{\zeta(2)}+O\left(\frac{1}{x}\right)\right)+O(x\log x),
\end{aligned}$$

by Example 4.2 on the tail end sum. ■

For the interested student we saw earlier as an example of Convolution Method I, that

$$\sum_{n\le x}\frac{\phi(n)}{n}=\frac{x}{\zeta(2)}+O(\log x).$$

Show that this also follows from Example 4.16 by Partial summation.

Problem 4.17 *Try the method to find a result for $\sum_{n\le x}\sigma(n)$ that is **not** by applying partial summation to $\sum_{n\le x}\sigma(n)/n$.*

Proof. Starting from $\sigma = 1 * j$ we have $\sigma(n) = \sum_{ab=n} b$ and so

$$\begin{aligned}\sum_{n\le x}\sigma(n) &= \sum_{n\le x}\sum_{ab=n} b = \sum_{ab\le x} b = \sum_{a\le x}\sum_{b\le x/a} b \\ &= \sum_{a\le x}\frac{1}{2}\left[\frac{x}{a}\right]\left(\left[\frac{x}{a}\right]+1\right) \\ &= \frac{1}{2}\sum_{a\le x}\left(\left(\frac{x}{a}\right)^2 + O\left(\frac{x}{a}\right)\right) \qquad (11)\end{aligned}$$

The first term here contains a convergent sum so use the idea in Convolution I, replace it by a sum over **all** a and estimate the error as in

$$\frac{1}{2}\sum_{a\le x}\left(\frac{x}{a}\right)^2 = \frac{1}{2}x^2\left(\sum_{a=1}^{\infty}\frac{1}{a^2} - \sum_{a>x}\frac{1}{a^2}\right) = \frac{1}{2}x^2\left(\zeta(2) + O\left(\frac{1}{x}\right)\right).$$

For the second term in (11) we have

$$\ll x\sum_{a\le x}\frac{1}{a} \ll x\log x.$$

Combine these results to get

$$\sum_{n\le x}\sigma(n) = \frac{1}{2}\zeta(2)\,x^2 + O(x\log x)\,.$$

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